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On the universality of matrix models for random surfaces

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Abstract. We present an alternative procedure to eliminate irregular contributions in the perturbation expansion of $c=0$–matrix models representing the sum over triangulations of random surfaces, thereby reproducing the results of Tutte [1] and Brézin et al. [2] for the planar model. The advantage of this method is that the universality of the critical exponents can be proven from general features of the model alone without explicit determination of the free energy and therefore allows for several straightforward generalizations including cases with non-vanishing central charge $c < 1$.

1 Introduction

The use of matrix models for the description and solution of theories of 2-dimensional quantum gravity coupled to matter with conformal weight $c \leq 1$ is well established (see e.g. the reviews [3] and references therein). Critical exponents agree with those found in the continuum description where methods of conformal quantum field theory can be applied [4]. The fact that this agreement is by no means trivial is mostly overlooked, although the equivalence of the theory of continuous 2-dimensional surfaces and the theory of abstract (combinatorical) triangulations with respect to their critical behaviour is based on many assumptions and indeed is presumably wrong for $c > 1$. For the case of pure gravity ($c = 0$) an integration over all metrics on a 2-dimensional surface modulo diffeomorphisms is replaced by a summation over abstract triangulations T, which are defined merely by the adjacency properties of their points:

$$Z_{\text{pure}}^{\text{cont}} = \int \frac{\mathcal{D}g_{\alpha\beta}}{diff}\, \mathrm{e}^{-S} \;\longrightarrow\; Z_{\text{pure}}^{\text{discr}} = \sum_{\mathrm{T}} \mathrm{e}^{-S} \;, \qquad (1)$$

with $S = \zeta\chi + \mu A$ in both cases if we identify χ with the continuous and discrete version of the Euler characteristic and A with the surface area and the number of triangles, respectively.

A second assumption enters, when one replaces the summation over abstract triangulations by a summation over Feynamn graphs. By a duality transformation each abstract triangulation can be identified with a Feynman graph of an $N \times N$ hermitean matrix model with cubic potential [5]. Not all Feynman graphs, however, correspond to regular triangulations. The universality of the corresponding two statistical ensembles is the subject of this letter. In a continuum limit, where $A \to \infty$, those graphs which from now on we refer to as *irregular*, even dominate over the regular ones (see below, (19)).

The concrete case of pure ($c=0$) quantum gravity corresponds to a 1-matrix model and the partition function for connected triangulations is given by the free energy as the generating functional for connected vacuum graphs:

$$Z_{\text{pure}}^{\text{discr}} \;\sim\; F_N^{\text{matrix}}(g) \equiv \frac{1}{N^2} \log Z_N^{\text{matrix}}(g) = \sum_{h,A} \mathcal{P}_h(A)\, g^A N^{-2h} \;, \qquad (2)$$

where

$$Z_N^{\text{matrix}}(g) = \int \mathrm{d}^{N^2}\Phi \, \exp\left(-\frac{1}{2}\mathrm{tr}\, \Phi^2 + \frac{g}{\sqrt{N}}\mathrm{tr}\, \Phi^3 \right) \;. \qquad (3)$$

Here A denotes the number of vertices of the graph and $\mathcal{P}_h(A)$ is the number of graphs with given A and genus h. As usual, (3) is to be understood as a formal representation of an asymptotic expansion in powers of g. Correspondingly, operations on such expressions (taking the logarithm, differentiation, integration, etc.) are operations on formal power series expansions. As (2) is a topological expansion in $\frac{1}{N}$ the limit $N \to \infty$ results in the *planar* model to which we want to restrict in the following.

The purpose of this letter is to show that universality of the planar cubic model can be proven without knowing details of the model, as e.g. the spectral distribution of the matrix eigenvalues in the limit $N \to \infty$, which was required for the results in [2], or the combinatorics of triangulations, as it was used in [1]. The idea is to introduce new couplings in the matrix model which can be adjusted using constraint equations such that the irregular contributions in the perturbation expansion cancel. Without explicitly solving these equations they can be used to relate the generating functional for regular graphs, $F^{\text{reg}}(g)$, to the generating functional for all graphs, $F^{\text{all}}(g) = F_\infty^{\text{matrix}}(g)$, thereby proving universality. Furthermore, given the behaviour of $F^{\text{all}}(g)$ close to its singularity, our method al-

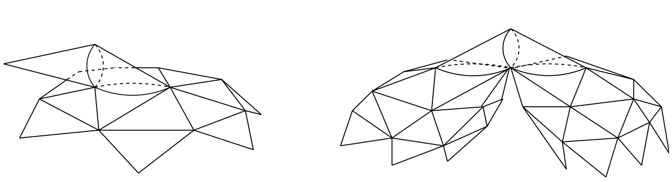

Fig. 1. Dual constructions of tadpole graphs

lows to determine the radius of convergence (i.e. the critical coupling) for $F^{\mathrm{reg}}(g)$.

We demonstrate this method for the case $c=0$, where the logic of the procedure can be illustrated most clearly. The extension to models with $0<c<1$ will in general be straightforward, and indeed has partly been used to prove universality for the case $c=\frac{1}{2}$ [6].

In Sect. 2 we classify those irregular Feynman graphs which do not correspond to triangulations. In Sect. 3 we relate the generating functional for regular graphs with the free energy of a modified matrix model with renormalized couplings by formulating constraint equations for these couplings. The proof of universality follows in Sect. 4. In Sect. 5 we make use of some known facts about the unregularized model to determine the radius of convergence, i.e. the critical point, for the regularized model.

2 Irregular graphs

The logic of our method is the construction of a *regularized* model by elimination of all irregular graphs and the subsequent direct comparison of its critical behaviour with that of the original one. The first step thus consists in identifying the irregular graphs.

Consider graphs containing 1-point and non-trivial 2-point subgraphs: In the dual picture these correspond to situations where either two vertices of the same triangle are identified or two vertices of two different triangles are identified without identification of the connecting edges (links) – see Fig. 1 and 2. Those configurations are forbidden in the context of triangulations as discrete 2-dim. manifolds, see e.g. [7]. In turn, these are also the only irregularities that can arise from the planar cubic model considered here, i.e. we have a one-to-one correspondence between irregular graphs and graphs containing tadpoles and/or non-trivial 2-point subgraphs.

Note that this argumentation is independent of the value of c. It depends, however, on the fact hat we are restricting to planar graphs (there exist non-planar graphs without tadpoles and non-trivial 2-point subgraphs which do not correspond to regular triangulations) and to graphs of valence 3.

3 Construction of the regularized model

The construction of the regularized model, i.e. the generating functional $F^{\mathrm{reg}}(g)$ for the numbers of regular trian-

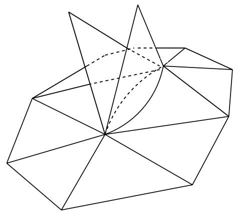

Fig. 2. General situation in the dual construction of a 2-point subgraph

gulations, can in general be achieved along the following steps:

1. Introduce a modified partition function (and corresponding free energy) from a matrix action, which contains general couplings for those contributions, on which one wants to put the contraints, i.e. 1-point- and 2-point-functions:
$$S^{\mathrm{mod}}(\Phi) = -\frac{\alpha}{2}\mathrm{tr}\,\Phi^2 + \frac{g}{\sqrt{N}}\mathrm{tr}\,\Phi^3 + \rho\sqrt{N}\mathrm{tr}\,\Phi$$
$$Z_N^{\mathrm{mod}}(g,\rho,\alpha) = \mathrm{e}^{N^2 F^{\mathrm{mod}}(g,\rho,\alpha)} = \int \mathrm{d}^{N^2}\Phi\, \mathrm{e}^{-S^{\mathrm{mod}}(\Phi)}, \tag{4}$$
2. Impose two conditions on the free energy of the modified model (4), where tadpoles are removed by setting the 1-point-function of the modified model equal to zero, i.e.
$$\frac{\partial F^{\mathrm{mod}}(g,\rho,\alpha)}{\partial\rho} = 0\,, \tag{5}$$
and self-energy contributions represented by non-trivial 2-point subgraphs are eliminated by assigning the value of the free propagator to the full 2-point function:
$$\frac{\partial F^{\mathrm{mod}}(g,\rho,\alpha)}{\partial\alpha} = -\frac{1}{2}\,. \tag{6}$$
A graphical representation of these two conditions is sketched in Fig. 3.
3. Evaluate the conditions (5) and (6) to find $\alpha(g)$ and $\rho(g)$. The free energy $F^{\mathrm{reg}}(g)$ is obtained from $F^{\mathrm{mod}}(g,\rho,\alpha)$ by a Legendre transformation with respect to ρ and α,
$$\begin{aligned} F^{\mathrm{reg}}(g) &= \\ &= \left[F^{\mathrm{mod}}(g,\rho,\alpha) - \frac{\partial F^{\mathrm{mod}}}{\partial\rho}\rho - \frac{\partial F^{\mathrm{mod}}}{\partial\alpha}\alpha\right]_{\rho=\rho(g),\alpha=\alpha(g)} \\ &= \left[F^{\mathrm{mod}}(g,\rho,\alpha) + \frac{1}{2}\alpha\right]_{\rho=\rho(g),\alpha=\alpha(g)}, \end{aligned} \tag{7}$$
where we have made explicit use of the conditions (5) and (6) in the second line. It will turn out that for the proof of universality it is not necessary to know F^{mod} or F^{all} explicitly.

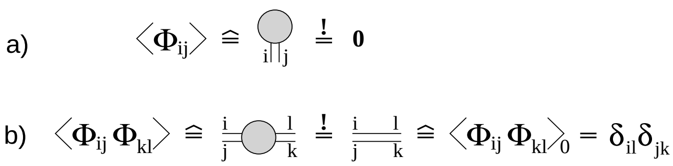

Fig. 3. Elimination of irregular graphs: a) Tadpole elimination b) Elimination of self-energy contributions

4 Proof of universality

For the proof of universality we have to show that $F^{\mathrm{all}}(g)$ and $F^{\mathrm{reg}}(g)$ exhibit the same critical exponents when g approaches its respective radius of convergence. Equivalently, we will prove that the derivatives of both functions with respect to g have the same critical behaviour. First we note that

$$\frac{\mathrm{d}F^{\mathrm{reg}}(g)}{\mathrm{d}g} \;=\; \left.\frac{\partial F^{\mathrm{mod}}(g,\rho,\alpha)}{\partial g}\right|_{\rho=\rho(g),\alpha=\alpha(g)} ,$$

which is a general consequence of (7). On the other hand, F^{mod} satisfies a generalized scaling relation which is easily obtained from (4) by a change of variables $\Phi \to \lambda\Phi$:

$$F^{\mathrm{mod}}(g,\rho,\alpha) \;=\; \ln\lambda + F^{\mathrm{mod}}(\lambda^3 g, \lambda\rho, \lambda^2\alpha) \; .$$

Differentiating with respect to λ, setting $\lambda = 1$, and inserting the constraint equations (5) and (6) leads to

$$\left.\frac{\partial F^{\mathrm{mod}}(g,\alpha,\rho)}{\partial g}\right|_{\rho=\rho(g),\alpha=\alpha(g)} \;=\; \frac{\alpha(g)-1}{3g} \; . \tag{8}$$

Thus, we have to show that $\alpha(g)$ has the same critical behaviour as $\partial F^{\mathrm{all}}(g)/\partial g$. For this we will derive a relation between F^{all} and F^{mod} and use the constraint equations to obtain a relation between α and the derivative of F^{all}.

The partition function (4) is not changed by the introduction of a new integration variable $\Phi = a\hat{\Phi} + b\mathbf{I}$ (where $\mathbf{I}$ denotes the $N \times N$ identity matrix). However, the two parameters a and b may be tuned such that the action expressed in terms of $\hat{\Phi}$ contains no linear term, and the quadratic term appears with a factor $-1/2$. In this way the modified model can be related to the original matrix model and we obtain:

$$F^{\mathrm{mod}}(g,\rho,\alpha) = \tag{9}$$
$$= F^{\mathrm{all}}(k) \;-\; \frac{1}{4}\,\log\left(\alpha^2 - 12\,g\rho\right) \;+\; \frac{\alpha-\sqrt{\alpha^2-12g\rho}}{6\,g} \;\times$$
$$\times\left[\rho + \frac{(\alpha-\sqrt{\alpha^2-12g\rho})^2}{36\,g} - \frac{\alpha(\alpha-\sqrt{\alpha^2-12g\rho})}{12\,g}\right] ,$$

where

$$k \equiv a^3 g \;=\; \frac{g}{(\alpha^2-12g\rho)^{\frac{3}{4}}} \; . \tag{10}$$

Inserting this relation into the conditions (5) and (6) leads to

$$\frac{3\,g}{\alpha^2-12g\rho}\left(1+3k\frac{\partial F^{\mathrm{all}}(k)}{\partial k}\right) + \frac{\alpha-\sqrt{\alpha^2-12g\rho}}{6\,g} = 0 \tag{11}$$

$$\frac{\alpha}{\alpha^2-12g\rho}\left(1+3k\frac{\partial F^{\mathrm{all}}(k)}{\partial k}\right) + \left(\frac{\alpha-\sqrt{\alpha^2-12g\rho}}{6\,g}\right)^2 = 1 \; . \tag{12}$$

From these two conditions we can eliminate $\partial F^{\mathrm{all}}(k)/\partial k$ to obtain the first solution

$$\rho(g) \;=\; -3\,g \; . \tag{13}$$

Without the explicit form of $F^{\mathrm{all}}(k)$ the remaining condition cannot be solved to obtain $\alpha(g)$. However, for the proof of universality we only need to confirm, that $\alpha(g)$ has the same critical behaviour for $g \to g_{\mathrm{c}}$ as $\partial F^{\mathrm{all}}(k)/\partial k$ for $k \to k_{\mathrm{c}}$. This follows immediately by inserting the solution $\rho = -3g$ into the remaining constraint equation, say (11), and making an expansion in $\delta k \equiv k - k_{\mathrm{c}}$. For the equation to hold, the leading non-integer power of $\delta g = g - g_{\mathrm{c}}$ in $\alpha(g)$ has to be equal to the leading non-integer power of δk in $\partial F^{\mathrm{all}}/\partial k$. This completes the proof of universality.

The whole procedure – and thus the proof of universality – immediately carries over to planar $c=0$ one-matrix-models with arbitrary even potential of the order $2p$. In these models tadpoles are absent and the remaining renormalization of the 2-point function leads to an expression analogous to (8), with the 3 in the denominator replaced by $2p$.

Furthermore, note that relations (7), (10) and (11) are independent of N. Furthermore, in the case of complex instead of hermitean matrices, condition (6) guarantees the elimination of non-trivial 2-point subgraphs for *arbitrary topologies*. In general however, there will exist other irregularities not stemming from non-trivial 2-point subgraphs whose systematic elimination fails because their classification is unclear.

5 Critical behaviour

We now want to calculate the radius of convergence, g_{c}, of the generating function of regular graphs, $F^{\mathrm{reg}}(g)$. This corresponds to the critical point of the regularized model. For this we have to know the radius of convergence, k_{c}, of $F^{\mathrm{all}}(k)$ as well as the leading coefficient a_1 in an expansion of $F^{\mathrm{all}}(k)$ around this critical point. We take these values from [2], for details of the calculation see also [8]:

$$k_{\mathrm{c}} = \sqrt{\frac{1}{108\sqrt{3}}} \tag{14}$$

$$a_1 \equiv \left.\frac{\partial F^{\mathrm{all}}(k)}{\partial k}\right|_{k=k_{\mathrm{c}}} = -\frac{2}{3\,k_{\mathrm{c}}}\left(5-3\sqrt{3}\right) \; . \tag{15}$$

We now evaluate (11) at the critical point. To lowest order we find

$$0 = \frac{1-3a_1k_{\rm c}}{\alpha_{\rm c}^2+36g_{\rm c}^2} + \frac{\alpha_{\rm c}-\sqrt{\alpha_{\rm c}^2+36g_{\rm c}^2}}{18\,g_{\rm c}^2}\,. \tag{16}$$

With the given values for $k_{\rm c}$ (14) and a_1 (15) we therefore obtain an equation between $g_{\rm c}$ and $\alpha_{\rm c} = \alpha(g_{\rm c})$. A second independent relation between these two quantities is provided by (10), evaluated at the critical point with the known solution $\rho = -3g$. From these two equations we immediately obtain the new critical point to be

$$g_{\rm c} = \sqrt{\frac{3}{256}}\,. \tag{17}$$

This value agrees with the result obtained by [2] and [1].

Let us finally add a comment on the fact, that universality is by no means trivial. From the known cricital values we can deduce that the number of graphs $n(A)$ as a function of the number of vertices A asymptotically grows like

$$n(A) \overset{A\to\infty}{\longrightarrow} \sim \left(\frac{1}{k_{\rm c}}\right)^A A^\kappa\,, \tag{18}$$

where κ is the critical exponent proven to be universal. $k_{\rm c}$ is the radius of convergence of the corresponding generating functional ($F^{\rm all}$ or $F^{\rm reg}$). (We should note that for our choice of the matrix action (3) the combinatorics of the perturbation expansion leads to a factor of 3 for each vertex, i.e. after a duality transformation one obtains an extra factor of 3 for each triangle in a triangulation. Our notation agrees with the one used by [2] and differs from [1] by this factor of 3 for each triangle.)

Therefore, the ratio of the number of regular graphs $n^{\rm reg}(A)$ to the number of all graphs $n^{\rm all}(A)$ for large values of A is given by

$$\frac{n^{\rm reg}(A)}{n^{\rm all}(A)} \overset{A\to\infty}{\longrightarrow} \sim \left(\frac{256}{3\cdot 108\sqrt{3}}\right)^{A/2} \overset{A\to\infty}{\longrightarrow} 0\,. \tag{19}$$

Thus, in the critical region the regular graphs considered as a subset of all graphs represent a partition of measure zero. So one cannot argue that universality holds because the regular graphs "dominate" the ensemble.

6 Summary and outlook

We presented a new and straightforward method of proving universality of the planar, cubic ($c=0$) matrix model with respect to the elimination of graphs not corresponding to regular triangulations, thereby reproducing results of Brézin et al. with, however, much less information needed about the original model. Our method also allows the determination of the new critical point from the knowledge of the old one.

An interesting generalization would be to models for which $c \geq 0$. The extension of our method to models with $0 \leq c \leq 1$ is in principle possible and has partly been used in [6] for the case $c = 1/2$. The situation for $c \geq 1$, however, is still unclear (see also the references in [9]).

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